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Liouville Transformation and Exactly Solvable Schrödinger Equations

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The present paper discusses the connection between exactly solvable Schrodinger equations and the Liouville transformation. This transformation yields a large class of exactly solvable potentials, including the exactly solvable potentials introduced by Natanzon. In addition, this class is shown to contain two new families of exactly solvable potentials.

1. INTRODUCTION

The study of exactly solvable Schrodinger equations dates back to the very beginnings of quantum mechanics. As examples one can site the harmonic oscillator, Coulomb, Morse (1929) Poschll–Teller (1933), Eckart (1930), and Manning–Rosen (1933) potentials. One can argue that in each of these cases the exact solvability comes about because the Schrodinger equations in question can be transformed by a gauge transformation and by a change of variables into either the hypergeometric or the confluent hypergeometric equation. To be more precise, in each of the above cases there exists a gauge factor $\sigma(z; E)$, which depends on the energy parameter E, and a change of variables $z = z(r)$, which does not, such that solutions to the corresponding Schrodinger equation,

$$-\psi''(r; E) + U(r)\psi(r; E) = E\psi(r; E) \tag{1}$$

are of the form

$$\psi(r; E) = \exp[\sigma(z(r); E)]\phi(z(r); E) \tag{2}$$

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where $\phi(z; E)$ is either $F(\alpha, \beta; \gamma; z)$, the Gauss hypergeometric function, or $\Phi(\alpha; \gamma; z)$, the confluent hypergeometric function, and where the parameters α, β, γ are themselves functions of E. The just mentioned types of special functions are well understood, and as a consequence one can explicitly calculate the bound state and scattering information for the corresponding potentials. In light of these remarks the following question is of interest.

Problem 1. Given a collection of functions $\mathcal{F} = \{\phi(z)\}$, find all possible potentials $U(r)$ such that there exist an E-dependent gauge factor $\sigma(z; E)$ and an E-independent change of variables $z(r)$ such that the solutions of equation (1) are of the form shown in (2).

For the cases of hypergeometric and confluent hypergeometric functions, Problem 1 was solved by Natanzon (1971). The corresponding classes of exactly solvable potentials have come to be known as Natanzon's hypergeometric and confluent hypergeometric potentials, and have been the subject of some discussion in the literature (Ginocchio, 1984; Cordero and Salamó, 1993; Wu *et al.*, 1989). The purpose of the present paper is to review Natanzon's approach and then to enlarge Natanzon's class of exactly solvable potentials by allowing ϕ to come from a larger class of special functions, namely the solutions of the following class of differential equations:

$$A(z)\phi''(z) + B(z)\phi'(z) + C\phi(z) = 0 \tag{3}$$

where $A(z)$ is a nonzero real polynomial of degree 2 or less, $B(z)$ is a real polynomial of degree 1 or less, and C is a real constant.

Prior to Natanzon, Problem 1 was considered by Bose (1964) as well as other authors (Manning, 1938; Bhattacharjie and Sudarshan, 1964). Bose's paper is noteworthy because it introduced the approach that was followed by Natanzon in his classification. This approach relies on two techniques: a certain canonical form for linear, second-order differential operators, and the Liouville transformation. The Liouville transformation will be described in Section 2 and the Bose–Natanzon approach in Section 3. The solution of Problem 1 for the case where $\mathcal{F}$ is the set of solutions of equation (3) is given in Section 4. The resulting collection of potentials includes Natanzon's hypergeometric and confluent hypergeometric potentials, as well as two new classes of exactly solvable potentials. These new potentials will be discussed in Section 5.

2. THE LIOUVILLE TRANSFORMATION

Consider a linear, second-order differential equation

$$a(z)\phi''(z) + b(z)\phi'(z) + c(z)\phi(z) = 0 \tag{4}$$

Dividing through by $a(z)$ and making the gauge transformation

$$\phi(z) = \exp\left(\int^z \frac{b(t)}{2a(t)}\, dt\right) \phi(z) \tag{5}$$

changes the equation into the following self-adjoint, canonical form:

$$\phi''(z) + I(z)\phi(z) = 0 \tag{6}$$

where the potential term is given by

$$I = \frac{1}{4a^2}\,(4ac - 2ab' + 2ba' - b^2) \tag{7}$$

Clearly, $I(z)$ is an invariant of equation (4) with respect to gauge transformations and multiplication by functions, and this is why equation (6) is being called a canonical form. Henceforth, $I(z)$ will be called the Bose invariant of equation (4).

A change of the independent variable, say $z = z(r)$, will transform equation (6) into

$$[z'(r)]^{-2}\, \tilde{\phi}''(r) - \frac{z''(r)}{[z'(r)]^3}\, \tilde{\phi}'(r) + I(z(r))\tilde{\phi}(r) = 0$$

where $\tilde{\phi}(r) = \phi(z(r))$. The corresponding canonical equation is

$$\psi''(r) + J(r)\psi(r) = 0 \tag{8}$$

where

$$\psi(r) = [z'(r)]^{-1/2}\tilde{\phi}(r) \tag{9}$$

$$J(r) = [z'(r)]^2\, I(z(r)) + \frac{1}{2}\,\{z, r\} \tag{10}$$

and where the term in curly brackets is the Schwarzian derivative of z with respect to r, namely

$$\{z, r\} = \left[\frac{z''(r)}{z'(r)}\right]' - \frac{1}{2}\left[\frac{z''(r)}{z'(r)}\right]^2$$

The above process of going from one self-adjoint equation to another by means of a change of variables has been named the Liouville transformation in Olver (1974) and the Liouville–Green transformation in Zwillinger (1992). The Liouville transformation arises naturally in the context of the WKB approximation (see Chapter 6 of Olver, 1974) and also underlies the following classical theorem due to Schwarz (see Hille, 1976, Theorem 10.1.1, or Olver, 1995, Theorem 6.28).

Theorem 2. The general solution to the Schwarzian equation

$$\{z, r\} = 2J(r)$$

has the form $z(r) = \psi_2(r)/\psi_1(r)$, where $\psi_2(r)$ and $\psi_1(r)$ are two linearly independent, but otherwise arbitrary solutions of equation (8).

In particular, this theorem implies that for every potential $J(r)$, there exists a change of variables $z(r)$ such that the corresponding Liouville transformation takes the equation $\phi''(z) = 0$ to equation (8). Therefore, one can relate any two equations of the form (8) by a Liouville transformation. It is for this reason that Problem 1 must be formulated with the condition that $z(r)$ not depend on the energy parameter. Without this restriction the problem would be uninteresting; one would get a criterion that would be satisfied by all possible potentials.

3. THE BOSE–NATANZON APPROACH

The approach in question rests on the following reformulation of Problem 1.

Problem 1A. Given a collection of functions $\mathcal{F} = \{\phi(z)\}$, find all possible $I_1(z) \geq 0$ and $I_0(z)$ such that for some E-dependent gauge factor $\sigma(z; E)$ the solutions of

$$\phi''(z; E) + [I_1(z)E + I_0(z)]\phi(z; E) = 0 \tag{11}$$

are of the form

$$\phi(z; E) = \exp(\sigma(z; E))\phi(z; E)$$

Indeed, suppose that I_1 and I_0 satisfy the above set of requirements. Let $z(r)$ be a solution of

$$z'(r) = (I_1(z))^{-1/2} \tag{12}$$

From formula (10) it follows that a Liouville transformation of equation (11) based on the change of variables $z = z(r)$ yields an equation with potential term $E - U(r)$, where

$$-U(r) = \frac{I_0(z)}{I_1(z)} + \frac{-4I_1(z)I_1''(z) + 5\,(I_1'(z))^2}{16(I_1(z))^3} \tag{13}$$

Furthermore, the corresponding eigenfunctions will have the form

$$\psi(r; E) = (I_1(z))^{1/4}\phi(z; E)$$

Therefore $U(r)$ satisfies the criterion imposed by Problem 1. One can also reverse the above argument to show that given a $U(r)$ demanded by Problem 1, one can produce an I_1 and an I_0 that satisfy the criterion of Problem 1A. In other words, the two formulations are equivalent.

The Bose invariant for the hypergeometric equation

$$z(1 - z)\phi''(z) + (\gamma - (1 + \alpha + \beta))\phi'(z) - \alpha\beta\phi(z) = 0$$

is given by

$$I(z) = \frac{T(z)}{4z^2(1 - z)^2}$$

where

$$T(z) = (1 - (\alpha - \beta)^2)z^2 + (2\gamma(\alpha + \beta - 1) - 4\alpha\beta)z + \gamma(2 - \gamma)$$

Note that every polynomial $T(z)$ of degree 2 or less can be obtained from some choice of α, β, γ. Therefore, in order to solve Problem 1A one must determine all possible $I_1(z)$ and $I_0(z)$ such that for all E there exists a $T(z; E)$ of degree two or less in z such that

$$I_1(z)E + I_0(z) = \frac{T(z; E)}{4z^2(1 - z)^2}$$

It is clear that this condition is satisfied if and only if $T(z; E) = R(z)E + S(z)$, where $R(z)$ and $S(z)$ are polynomials of degree 2 or less, and such that $R(z) \geq 0$ in the domain of interest. The determining relation for $z(r)$ follows from (12); it is

$$z'(r) = \frac{2z(1 - z)}{\sqrt{R(z)}}$$

Setting $R(z) = r_2 z^2 + r_1 z + r_0$, calculating $\{z, r\}$, and plugging the result into (10), one obtains the formula for Natanzon's hypergeometric potentials:

$$U = \frac{-S(z) + 1}{R(z)} + \left(\frac{r_1 - 2(r_2 + r_1)z}{z(1 - z)} - \frac{5}{4}\frac{(r_1^2 - 4r_2 r_0)}{R(z)} + r_2\right)\frac{z^2(1 - z)^2}{R(z)^2} \tag{14}$$

These potentials describe the solution of Problem 1 for the case where $\mathcal{F}$ is the set of hypergeometric functions.

It is well known that one can transform hypergeometric functions into confluent hypergeometric ones by a certain limit process. Natanzon obtained his confluent hypergeometric potentials by applying this limit process to his hypergeometric potentials. The resulting family of potentials can also be considered as a solution of Problem 1, but the corresponding $\mathcal{F}$ is not the set of confluent hypergeometric functions, but rather the set of *scaled* confluent hypergeometric functions; namely $\phi(z) = \Phi(\alpha; \gamma; \omega z)$, where ω is an extra scaling parameter. These functions satisfy the following scaled version of the confluent hypergeometric equation:

$$z\phi''(z) + (\gamma - \omega z)\phi'(z) - \omega\alpha\phi(z) = 0$$

The corresponding Bose invariant is

$$I(z) = \frac{-\omega^2 z^2 + 2\omega(\gamma - 2\alpha)z + \gamma(2-\gamma)}{4z^2}$$

where again every possible second-degree polynomial can occur in the numerator as one varies α, γ, ω. Hence, by the same reasoning as above, the criterion of Problem 1A will be satisfied if and only if

$$I_1(z)E + I_0(z) = \frac{R(z)E + S(z)}{4z^2}$$

where $R(z)$ and $S(z)$ are polynomials of degree 2 or less, such that $R(z) \geq 0$ in the domain of interest. The determining relation for $z(r)$ follows from (12); it is

$$z'(r) = \frac{2z}{\sqrt{R(z)}}$$

Setting $R(z) = r_2 z^2 + r_1 z + r_0$, calculating $\{z, r\}$, and plugging the result into (10), one obtains the formula for Natanzon's confluent hypergeometric potentials:

$$U = \frac{-S(z) + 1}{R(z)} + \left(\frac{r_1}{z} - \frac{5}{4}\frac{r_1^2 - 4r_2 r_0}{R(z)} - r_2\right)\frac{z^2}{R(z)^2} \tag{15}$$

4. GENERALIZED NATANZON POTENTIALS

The present section is devoted to the solution of Problem 1 for the case where $\mathcal{F}$ is the set of solutions $\phi(z)$ of equations of type (3). The Bose invariant for equation (3) is given by

$$I(z) = \frac{T(z)}{A(z)^2}$$

where $T(z)$ is a polynomial of degree 2 or less determined by quadratic combinations of the coefficients of $A(z)$, $B(z)$, and C. The exact formula for $T(z)$ is not important; it can be readily recovered from equation (7). What is significant is that for a fixed $A(z)$, one can obtain every possible $T(z)$ of degree 2 or less from some choice of $B(z)$ and C. Consequently, using the reformulation given by Problem 1A, one must seek all possible $I_1(z)$ and $I_0(z)$ such that for all values of E, there exist $T(z; E)$ and $A(z; E)$ both of degree 2 or less, such that

$$I_1(z)E + I_0(z) = \frac{T(z; E)}{A(z; E)^2} \tag{16}$$

Lemma 3. Suppose that for all E there exist $T(z; E)$ and $A(z; E)$ such that relation (16) holds. Then, there exist E-independent polynomials $A(z)$, $R(z)$, $S(z)$ of degree two or less such that $I_1 = R/A^2$ and $I_0 = S/A^2$.

Proof. Setting $E = 0$ in (16), one infers that $I_0 = T_0/A_0^2$, where the degrees of T_0 and A_0 are two or less. Similarly, by setting $E = 1$, one infers that $I_1 = T_0/A_0^2 - T_1/A_1^2$, where the degrees of T_1 and A_1 are two or less. Thus, relation (16) may be rewritten as

$$\frac{ET_1}{A_1^2} + \frac{(1 - E)T_0}{A_0^2} = \frac{ET_1 A_0^2 + (1 - E)T_0 A_1^2}{A_0^2 A_1^2} = \frac{T(z; E)}{A(z; E)^2} \tag{17}$$

Given rational functions P_0/Q_0 and P_1/Q_1 where the numerators and denominators are relatively prime polynomials, it is easy to show that the denominator of the reduced form of

$$\frac{P_0}{Q_0} + \lambda \frac{P_1}{Q_1}, \qquad \lambda \in \mathbb{C}$$

is the least common multiple of Q_0 and Q_1 for all but a finite number of λ values. This observation implies that the least common multiple of A_1 and A_0 must have degree less than or equal to 2.

The rest of the proof will be done by cases, based on the degree of A_1 and A_0. If both A_0 and A_1 are constants, then there is nothing to prove.

Next, consider the case where one of A_0 and A_1 is a constant, but the other one is not. Without loss of generality suppose it is A_1 that is constant. Then T_1 must be constant also, for otherwise the linear combinations of $T_0 A_1^2$ and $T_1 A_0^2$ would not always yield polynomials of degree 2 or less. Therefore the lemma is true for this case also.

Suppose next that both A_1 and A_0 have degree 1. If A_1 and A_0 have the same root, then there is nothing to prove. If A_1 and A_0 have different roots, then both T_1 and T_0 must be constants, because otherwise linear combinations of $T_0 A_1^2$ and $T_1 A_0^2$ would not always yield polynomials of degree 2 or less. Hence one can write

$$I_1 = \frac{T_1 A_0^2}{A_0^2 A_1^2}, \qquad I_0 = \frac{T_0 A_1^2}{A_0^2 A_1^2}$$

and this proves the lemma for the case under consideration.

Finally, suppose that one or both of A_1 and A_0 is second degree. Without loss of generality assume that A_1 has degree 2. Consequently, A_1 must be the least common multiple of A_0 and A_1, i.e., A_0 must be a factor of A_1. Now A_0 cannot be a constant, because generically, the degree of linear combinations of $T_1 A_0^2$ and $T_0 A_1^2$ would be greater than 2. If the degree of A_0 is 2, then there is nothing to prove. The last possibility is that $A_1 = \Delta A_0$, where both A_0 and Δ have degree 1. In this case

$$I_1 E + I_0 = \frac{ET_1 + (1 - E)\Delta^2 T_0}{A_1^2}$$

and hence T_0 must be a constant. But one can therefore write

$$I_0 = \frac{T_0 \Delta^2}{A_1^2}$$

i.e., the lemma is also true for this last case. ■

Using the above lemma as well as the formulas (12) and (13), one can now give the solution of Problem 1 for the case where $\mathcal{F}$ is the set of solutions to equation (3). The desired potentials have the form

$$U(r) = \frac{-S(z) + \mathcal{D}(A)/4}{R(z)} + \left(-\frac{3R''(z)}{2} - \frac{5}{4}\frac{\mathcal{D}(R)}{R(z)} + \frac{R'(z)A'(z)}{A(z)} \right) \frac{A(z)^2}{4R(z)^2} \tag{18}$$

where $A(z)$, $R(z)$, $S(z)$ are polynomials of degree two or less, $\mathcal{D}$ denotes the discriminant operator, and $z(r)$ is a solution of

$$z'(r) = \frac{A(z)}{\sqrt{R(z)}} \tag{19}$$

Henceforth this class of potentials will be referred to as the generalized Natanzon potentials.

5. NEW EXACTLY SOLVABLE POTENTIALS

Note that the presentation of the generalized Natanzon potentials given by equations (18) and (19) is invariant under affine substitutions, $z \mapsto az + b$. Consequently, no generality will be lost if one restricts $A(z)$ to one of the following five possibilities: 1, z, z^2, $z(z - 1)$, $z^2 + 1$. The second and the fourth cases yield, respectively, the Natanzon confluent hypergeometric and the Natanzon hypergeometric potentials. If $A(z) = z^2$, then the substitution $z \mapsto 1/z$ transforms (18) and (19) into the corresponding forms for the case of $A(z) = z$. Therefore, if $A(z) = z^2$, one again obtains the Natanzon confluent hypergeometric potentials. With a bit of work one can check that at the level of solutions to the respective forms of equation (3), this transformation corresponds to the to the well-known identity (see Chapter 6.6 of Erdélyi and Bateman, 1953)

$$ {}_2F_0(\alpha, \alpha + 1 - \gamma; 1/z) = z^{\alpha}\Psi(\alpha, \gamma; z) $$

where Ψ is the confluent hypergeometric function of the second kind.

The two remaining cases, $A(z) = 1$ and $A(z) = 1 + z^2$, yield new families of exactly solvable potentials. These will be referred to, respectively, as case 1 and case 5 potentials, and will now be examined in in some detail. In each case it will be convenient to rewrite equation (3) using a certain choice of adapted parameters. These adapted parameters will be denoted by Greek letters, and the resulting equation will be referred to as the primary equation.

The solutions of the primary equations can be given in terms of hypergeometric functions, and have a natural dependence on the adapted parameters. The actual potential depends on a choice of $R(z)$. The potential parameters—these will be denoted using lowercase Latin letters—and the energy parameter E will turn out to be related to the adapted parameters by polynomial relations. It is important to note that for fixed potential parameters and a fixed value of E one must solve these relations in order to obtain the corresponding values of the adapted parameters of the primary equation.

An examination of formula (18) shows that in order for $U(r)$ to be nonsingular, the z domain of the function shown in the right-hand side of (18) must not contain any roots of $R(z)$. Singular potentials will not be discussed here, and this constraint greatly reduces the possible choices for $R(z)$.

According to equation (19), the physical coordinate of the corresponding Natanzon potential is given by

$$ r(z) = \int^z \frac{\sqrt{R(t)}}{A(t)}\, dt $$

In all cases one can explicitly calculate the above antiderivative, but the inverse, i.e., $z(r)$, cannot in general be specified explicitly. Indeed, one may reasonably speculate that historically the study of Natanzon potentials was delayed by the fact that the inverse $z(r)$ of the above antiderivative can be given in terms of elementary functions only for certain restricted choices of $R(z)$, and $A(z)$ [based on the remarks found in the first paragraph of Natanzon (1971), it would seem that Natanzon shares this viewpoint].

Nonetheless, a great deal of information about the inverse is available. First, one can always take the domain of $z(r)$ to be the whole real line; this is a consequence of the fact that we excluded those $R(z)$ that have roots in the domain of z. One can calculate power series and asymptotic expansions for $z(r)$ and use these as the basis for a numerical approximation. The graphs of potential curves that are given below were generated using this approach.

In the subsequent discussion Φ and Ψ will denote the confluent hypergeometric functions of the first and second kind, and F will denote the usual hypergeometric function ${}_2F_1$. For the source of this notation as well as the various properties of these functions the reader is referred to Erdélyi and Bateman (1953).

Case 1. $A(z) = 1$

Primary Equation: $\phi''(z) - 2\omega(z-\beta)\phi'(z) - 4\alpha\omega\,\phi(z) = 0$

Primary Solutions:

$$\phi_0 = \Phi(\alpha; 1/2; \omega(z-\beta)^2), \qquad \phi_1 = (z-\beta)\,\Phi\,(\alpha + 1/2; 3/2; \omega(z-\beta)^2)$$

Note that the family of potentials under discussion is invariant under substitutions of the form $z \mapsto z + k$, $k \in \mathbb{R}$. In order to obtain a nonsingular potential, $R(z)$ must not have real roots, and consequently it is sufficient to consider the case $R(z) = z^2 + 1$. The most general form of the potential can then be obtained by a scaling transformation.

Distance 1-Form and Physical Variable:

$$dr = \sqrt{z^2+1}\,dz \qquad r = \frac{1}{2}\left(z\sqrt{z^2+1} + \log(z + \sqrt{z^2+1})\right)$$
$$z \sim \pm\sqrt{2|r|}, \qquad r \to \pm\infty \tag{20}$$

Potential:

$$U = \frac{az+b}{z^2+1} - \frac{3/4}{(z^2+1)^2} + \frac{5/4}{(z^2+1)^3}$$
$$E = -\omega^2, \qquad a = -2\beta\omega^2, \qquad b = \omega^2(\beta^2-1) - \omega(1-4\alpha) \tag{21}$$

The resulting two-parameter family of potentials is characterized by the presence of two wells separated by a barrier (see Fig. 1). The parameter a controls the degree of asymmetry; if $a = 0$, the potential is symmetric about $r = 0$. The parameter b controls the height of the central barrier. As b

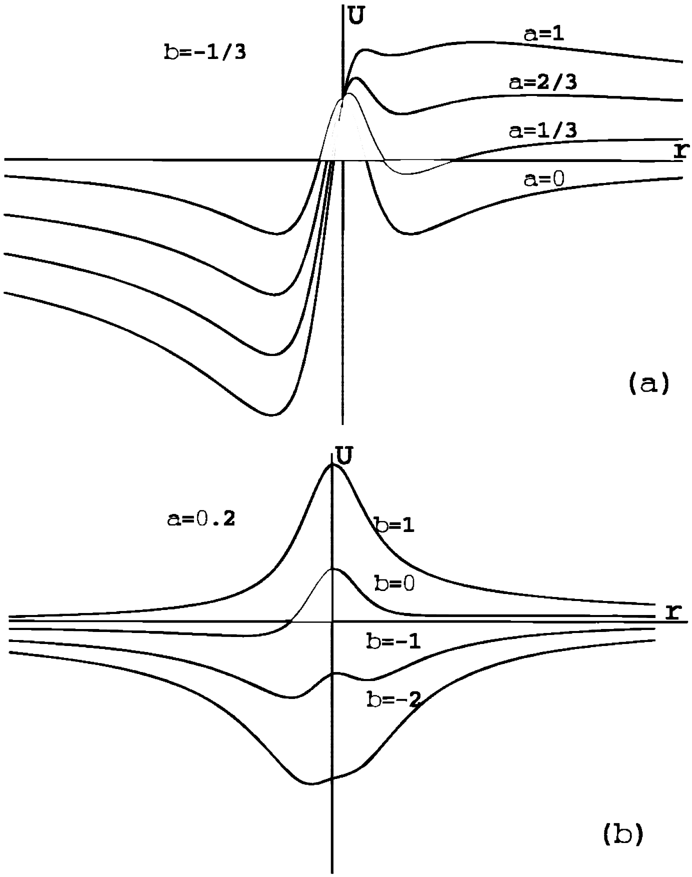

Fig. 1. Case 1: roles of the a and b parameters.

increases, the wells become smaller; if in addition $a = 0$, then they disappear altogether. As b decreases, the two wells merge into one.

Eigenfunctions:

$$\psi_i = e^{-1/2\omega(z-\beta)^2}(z^2+1)^{1/4}\,\phi_i, \qquad i = 0, 1 \tag{22}$$

Bound States. A bound state occurs when $-2\alpha \in \mathbb{N}$, and when $\omega > 0$. This directly implies that there are infinitely many bound states if $a \neq 0$ or if $b < 0$, and that there are no bound states otherwise.

Scattering. The scattering states occur when $E > 0$. Correspondingly, $\omega = i\tilde{\omega}$, and $\alpha = \frac{1}{4} + i\tilde{\alpha}$, where $\tilde{\omega} > 0$ and $\tilde{\alpha}$ are real. As a consequence

ψ_0 and ψ_1 are real-valued functions; this follows directly from the well-known formula (see Chapter 6.3 of Erdélyi and Bateman, 1953)

$$\Phi(a;\, c;\, z) = e^z \Phi(c - a;\, c;\, -z)$$

An asymptotically free eigenfunction, call it ψ_f, can be given explicitly in terms of the confluent hypergeometric function of the second kind:

$$\psi_f = \omega^\alpha e^{-1/2\omega(z-\beta)^2}(z^2 + 1)^{1/4}\Psi(\alpha;\, 1/2;\, \omega(z - \beta)^2)$$

From the well-known asymptotic formula (see Chapter 6.13 of Erdélyi and Bateman, 1953)

$$\Psi(a;\, c;\, z) \sim z^{-a}, \qquad z \to +\infty$$

and from (20) it follows that

$$\psi_f \sim \exp(-i\omega|r|) \exp\left[i\left(\pm\frac{a}{\omega^2\sqrt{2}}\sqrt{|r|} - \frac{a^2}{4\omega^3} \right)\right](\sqrt{2|r|} \mp \beta)^{2i\hat{\alpha}}$$

as $r \to \pm\infty$. Thus, asymptotically ψ_f represents an almost free particle traveling toward the center. The discontinuity in the direction of motion is caused by the fact that $\Phi(z)$ is not regular at $z = 0$. The extra terms in the asymptotic phase appear because of the slow rate—on the order of r^{-1} or $r^{-1/2}$, depending respectively on whether a is zero or not—at which the potential falls off toward zero.

From the relation between confluent hypergeometric functions of the first and second kinds (see Chapter 6.7 of Erdélyi and Bateman, 1953), one obtains

$$\psi_0 = c_0\psi_f + \overline{c_0\psi_f}, \qquad \psi_1 = \mathrm{sgn}(z)(c_1\psi_f + \overline{c_1\phi_f}).$$

where

$$c_0 = (-\omega)^{-\alpha}\frac{\Gamma(1/2)}{\Gamma(1/4 - i\alpha)}, \qquad c_1 = (-\omega)^{-1/2-\alpha}\frac{\Gamma(3/2)}{\Gamma(3/4 - i\alpha)}$$

It immediately follows that

$$\frac{1}{2}\left(\frac{\psi_0}{c_0} - \frac{\psi_1}{c_1}\right) = \begin{cases} T\psi_f, & z \to +\infty \\ \psi_f + R\bar{\psi}_f, & z \to -\infty \end{cases}$$

where the reflection and transmission coefficients are given by

$$T = e^{i\theta}(1 - e^{-2\alpha\pi i})^{-1}, \qquad R = e^{i\theta}(1 - e^{2\alpha\pi i})^{-1}$$

where

$$e^{i\theta} = \frac{\Gamma(1/2 - \alpha)}{\Gamma(\alpha)} E^{(\alpha - 1/4)} e^{-\pi i/4}$$

Case 5. $A(z) = z^2 + 1$

Primary Equation:

$$(1 + z^2)\phi''(z) + (i(1 - 2\rho) + (2\sigma + 1)z)\phi'(z) + (\sigma^2 - \delta^2)\phi = 0$$

The above equation is related to the usual hypergeometric equation,

$$\zeta(1 - \zeta)\phi_{\zeta\zeta} + (\gamma - \zeta(\alpha + \beta + 1))\phi_\zeta - \alpha\beta\phi = 0$$

by a linear change of parameters, and a complex-linear change of coordinates:

$$\sigma = \frac{\alpha + \beta}{2}, \qquad \delta = \frac{\alpha - \beta}{2}, \qquad \rho = \gamma - \frac{\alpha + \beta}{2}, \qquad \zeta = \frac{1 - iz}{2}$$

Primary Solutions:

$$\phi_1 = F(\sigma + \delta, \sigma - \delta; \rho + \sigma; (1 - iz)/2)$$
$$\phi_2 = F(\sigma + \delta, \sigma - \delta; 1 - \rho + \sigma; (1 + iz)/2)$$

To obtain nonsingular potentials one must take $R(z)$ without any real roots. The resulting family of potentials depends of four parameters. A treatment of the most general potential, i.e., one depending all four parameters, would be unduly long, and not particularly illuminating. Thus, the focus here will on a more manageable three-parameter subclass, namely the potentials that correspond to $R(z) = z^2 + a^2$.

Distance 1-Form and Physical Variable:

$$dr = \frac{\sqrt{z^2 + a^2}}{z^2 + 1}\, dz,$$
$$r = \sinh^{-1}\left(\frac{z}{a}\right) + \sqrt{1 - a^2}\, \tanh^{-1}\left(\frac{\sqrt{1 - a^2}z}{\sqrt{z^2 + a^2}}\right)$$
$$z \sim a \exp(\sqrt{1 - a^2}\, \tanh^{-1}(\sqrt{1 - a^2}))\sinh(r), \qquad r \to \pm\infty$$

Potential:

$$U = \frac{b + cz}{a^2 + z^2} + \frac{1}{4}\frac{z^2 + 1}{z^2 + a^2} + \left(\frac{z^2 + 1}{z^2 + a^2}\right)^2 \left(-\frac{1}{4} - \frac{1}{2}\frac{a^2 + 1}{z^2 + 1} + \frac{5}{4}\frac{a^2}{z^2 + a^2}\right) \quad (23)$$

$$E = -\delta^2, \qquad b = \delta^2(1 - a^2) - \left(\rho - \frac{1}{2}\right)^2 - \left(\sigma - \frac{1}{2}\right)^2$$
$$c = -2i\left(\rho - \frac{1}{2}\right)\left(\sigma - \frac{1}{2}\right) \tag{24}$$

These potentials fall off exponentially toward zero for large r. Setting $\rho = 1/2$, one obtains potentials that coincide with a certain subclass of Natanzon hypergeometric potentials. The correspondence is given by the following substitution: $z \mapsto 1 + z^2$, and at the level of solutions to the respective primary equations is described by the following quadratic transformation of the hypergeometric function (see Chapter 2.11 of Erdélyi and Bateman, 1953):

$$F(\sigma + \delta, \sigma - \delta; 1/2 + \sigma; (1 - iz)/2)$$
$$= F((\sigma + \delta)/2, (\sigma - \delta)/2; 1/2 + \sigma; 1 + z^2)$$

The generic shape is that of two spikes for $a > 1$, and two wells for $0 < a < 1$; when $a = 1$ one recovers a modified Poschll–Teller potential. The parameter b controls the height/depth of the central spike/well, while c is the skew parameter that controls the degree of asymmetry in the potential.

The case $a > 1$ results in the more interesting potential shapes, and thus will be the focus of the remaining discussion. Consider the symmetric potentials ($c = 0$) for a fixed value of $a > 1$. The number of extrema in the potential curve depends on the value of b. There are three critical values of b where the number of extrema changes:

$$\frac{7a^2 - 3 + 3a^{-2}}{20}, \qquad \frac{-a^2 + 9 - 9a^{-2}}{4}, \qquad -a^2 + \frac{3}{4}$$

At these critical values of b some of the extrema merge, and one obtains some distinguished potential shapes; these shapes are shown in Fig. 2a.

At the value $b = (-a^2 + 9 - 9a^{-2})/4$ the potential takes the form

$$\frac{a^2 - 1}{4a^4}\left(\frac{z^4((a^2 - 7)z^2 + 6a^4 - 12a^2)}{(z^2 + a^2)^3} - (a^2 - 7)\right)$$

One can show that $z = r/a + O(r^3)$ near $r = 0$, and hence the first three r-derivatives of the potential vanish. As a consequence, one obtains a well with a very flat bottom. The potential also possesses two local maxima; these correspond to spikelike barriers on either side of the well. The variation of a in this type of potential shape is shown in Fig. 2b. The two barriers vanish precisely at the critical value of $b = -a^2 + 3/4$.

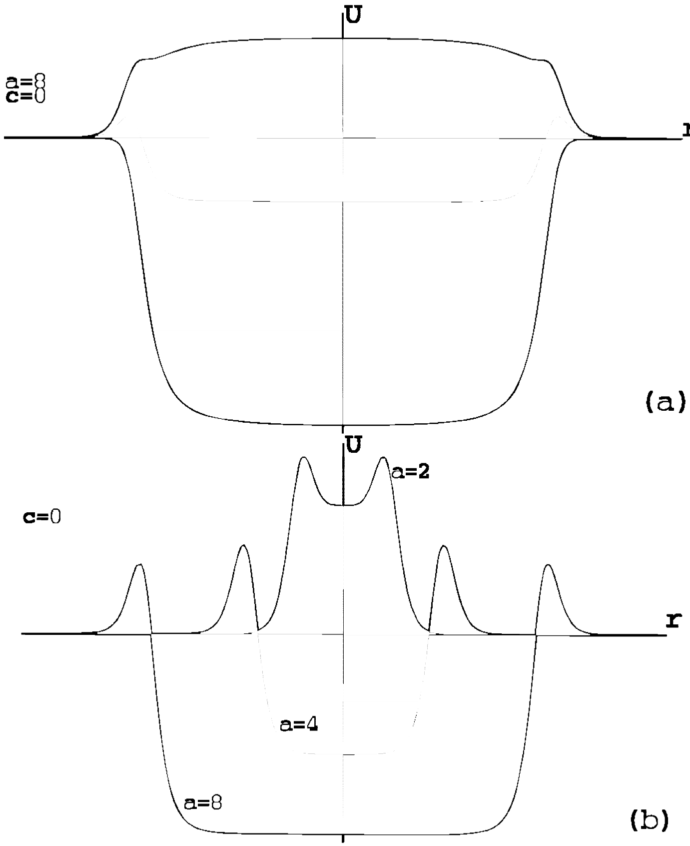

Fig. 2. Case 5 symmetric potentials: (a) critical values of the b parameter, (b) variation of the a parameter in potentials with the middle critical b value.

For $c \neq 0$ one can obtain similarly distinguished potentials whenever b attains a critical value where the potential extrema merge. There is no exact formula for these critical values of b; they must be solved for numerically. The resulting asymmetric potentials and their symmetric counterparts are shown in Fig. 3.

Eigenfunctions:

$$\psi_i = (1 - iz)^{\sigma/2+\rho/2-1/4} (1 + iz)^{\sigma/2-\rho/2+1/4} \left(\frac{a^2 + z^2}{1 + z^2}\right)^{1/4} \phi_i, \quad i = 1, 2 \qquad (25)$$

Bound States: An examination of relations (24) will show that in order to obtain a potential with real coefficients, δ must either be real

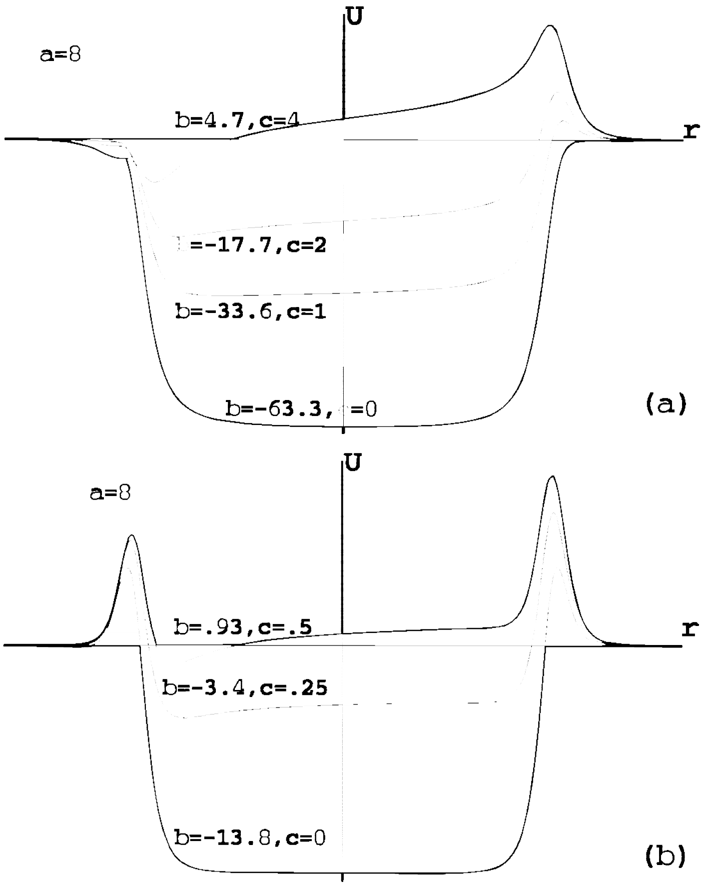

Fig. 3. Case 5 asymmetric potentials: variation of the c parameter in potentials with (a) lowest critical b value, (b) middle critical b value.

or imaginary; and either $\sigma - 1/2$ or $\rho - 1/2$ must be real, while the other must be imaginary.

Note that the following transformation of the parameters is a symmetry of the potential:

$$\rho \mapsto i(\sigma - 1/2) + 1/2, \qquad \sigma \mapsto -i(\rho - 1/2) + 1/2$$

The transformation $\delta \to -\delta$ is also a potential symmetry. The presence of these two symmetries means that without loss of generality one can assume that σ is real, that $\Re(\rho) = 1/2$, and that δ is either positive or positive imaginary. With these assumption in place, ψ_2 is the complex conjugate of ψ_1, and the latter is square integrable if and only if $\delta > 0$ and $\sigma + \delta \in -\mathbb{N}$.

After a bit of calculation one can show that this criterion implies that for fixed a, b, c, the bound states are indexed by natural numbers $N = -(\sigma + \delta)$ such that

$$N < \frac{\sqrt{-2b + 2\sqrt{b^2 + c^2}} - 1}{2}$$

In particular, if $c = 0$ (the symmetric potentials) then there will be no bound states if $b \geq -1/4$. If $b < -1/4$, then the number of bound states is equal to the largest integer smaller than $1/2 + \sqrt{-b}$.

Scattering. The scattering states occur when $E > 0$, and hence without loss of generality δ is positive imaginary. For reasons detailed above, σ will be assumed to be real, while $\Re(\rho)$ will be assumed to be 1/2. To compute the reflection and transmission coefficients it will be useful to introduce two more solutions of the primary equation:

$$\phi_3 = ((iz - 1)/2)^{-\sigma-\delta}\, F(\sigma + \delta, \delta + 1 - \rho; 1 + 2\delta; 2/(1 - iz))$$
$$\phi_4 = ((iz - 1)/2)^{-\sigma-\delta}\, F(\sigma - \delta, \rho - \delta; 1 - 2\delta; 2/(1 + iz))$$

As per the formula in (25), let ψ_3 and ψ_4 denote the corresponding eigenfunctions. The usefulness of ψ_3 and ψ_4 is that they represent asymptotically free particles traveling, respectively, toward and away from the origin:

$$\psi_3 \sim K^{-\delta}\, e^{\mp(\sigma+\rho+\delta-1/2)\,\pi i/2}\, e^{-\delta|r|}, \qquad r \to \pm\infty$$
$$\psi_4 \sim K^{\delta}\, e^{\mp(\sigma+\rho-\delta-1/2)\pi i/2}\, e^{\delta|r|}, \qquad r \to \pm\infty$$

where

$$K = \frac{a}{4} \exp\left(\sqrt{1 - a^2}\, \tanh^{-1}\left(\sqrt{1 - a^2}\right)\right)$$

Relations between the regular eigenfunctions ψ_1, ψ_2 and the irregular ones ψ_3, ψ_4 are given by (see Chapter 2.9 of Erdélji and Bateman, 1953)

$$\psi_1 = c_3\psi_3 + c_4\psi_4, \qquad \psi_2 = \overline{c_4} e^{\pm\pi i(\sigma+\delta)}\psi_3 + \overline{c_3} e^{\pm\pi i(\sigma-\delta)}\psi_4$$

where the $\pm$ in the second equation corresponds to the sign of z, and where

$$c_3 = \frac{\Gamma(\sigma + \rho)\Gamma(2\delta)}{\Gamma(\rho + \delta)\Gamma(\sigma + \delta)}, \qquad c_4 = \frac{\Gamma(\sigma + \rho)\Gamma(-2\delta)}{\Gamma(\rho - \delta)\Gamma(\sigma - \delta)}$$

It follows that

$$K^{\delta} e^{-(\sigma+\delta+\rho-1/2)\pi i/2}\left(\frac{\psi_1}{c_3} e^{\pi i(\sigma+\delta)} - \frac{\psi_2}{c_4}\right)$$

$$= \begin{cases} Te^{\delta r}, & r \to +\infty \\ e^{\delta r} + Re^{-\delta r}, & r \to -\infty \end{cases}$$

where elementary calculations show that

$$T = \frac{K^{2\delta}\Gamma(\sigma-\delta)\,\Gamma(1-\sigma-\delta)\Gamma(\rho-\delta)\Gamma(1-\rho-\delta)}{2\pi\Gamma(-2\delta)\Gamma(1-2\delta)}$$

$$R = T\left(\frac{\sin(\pi\sigma)\,\sin(\pi\rho)}{\sin(\pi\delta)} - \frac{i\cos(\pi\sigma)\,\cos(\pi\rho)}{\cos(\pi\delta)}\right)$$

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